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THE CHEBYSHEV RIDGE POLYNOMIALS IN 2D TENSOR TOMOGRAPHY

S. G. KAZANTSEV[1] and A. A. BUKHGEIM[2]

[1]*Sobolev Institute of Mathematics, Siberian Branch of Russian Academy of Sciences, Acad. Koptyug prosp., 4, Novosibirsk, 630090, Russia*
e-mail: kazan@math.nsc.ru
[2]*Schlumberger, Aslakveien 14A, 0753 Oslo, Norway*
e-mail: ABoukhgueim@slb.com

Abstract - The approach to the study of vector and 2-tensor fields tomography problems on the plane is presented. The new orthogonal polynomial bases of vector and symmetrical 2-tensor solenoidal fields were built with the help of bivariate Chebyshev ridge polynomials. These bases are useful not only in tomography, but also have potential applications in fluid mechanics, electromagnetism and image processing problems. This approach can be generalized for the m-tensor field tomography of arbitrary rank m. The numerical results of this novel inversion algorithm for vector and tensor Radon transform are presented.

1. INTRODUCTION

In vector tomography we need to reconstruct a vector field from the tomographic measurements, which are often modeled by *the inner product probe transform* of a vector field [12], [19]. This transform is the inner product of the classical Radon transform of a vector field with a unit vector, called *the probe*, which may be a function of the projection orientation. The special cases of the inner product probe transform are *the longitudinal* and *transverse transforms*, also known as *the vectorial Radon transform* and *Radon normal transform* respectively. In this paper we deal with the longitudinal transform (vectorial Radon transform), which appears in acoustic transmission measurements (time-of-flight data) and in ultrasound Doppler backscattering measurements on (blood) flows, namely the first moment of the spectra velocity is interpreted by means of vectorial Radon transforms, see [3], [11], [15], [18]. The tensorial Radon transform arise in integrated photoelasticity, when one needs to reconstruct the optical and stress tensor fields in the elastic and transparent media from optical transmission measurements (interferometric measurements), see [1], [17]. Many more facts about tensor tomography problems and other references to this theme can be found in [1], [6], [10], [13], [16] and [17].

In this paper we consider the tensor tomography problem of the object space $\mathbf{L}_2(\mathbb{D}, \mathbf{S}_m)$ — the space of symmetrical 2D tensor fields of rank $m = 1$ or $m = 2$ and square integrable in the unit disc $\mathbb{D}$. We construct a new orthonormal polynomial basis for $\mathbf{L}_2(\mathbb{D}, \mathbf{S}_m)$, that was built with the help of bivariate Chebyshev ridge polynomials. It is known that only the solenoidal (divergence-free) part of vector/tensor field can be reconstructed from vector/tensor Radon transform, thus the orthogonal polynomial bases was constructed both for solenoidal and potential (irrotational) subspaces of $\mathbf{L}_2(\mathbb{D}, \mathbf{S}_m)$. The novel inversion algorithm for the vector and 2-tensor fields tomography problems was developed and numerically implemented.

Our previous work [9] deals with the same tensor tomography problems in the fan beam scanning geometry that were solved with the help of Zernike polynomials.

1.1. Statement of the Vector and 2-Tensor Fields Tomography Problems

Let us consider the Cartesian coordinate system (x^1, x^2) on the plane $\mathbf{R^2}$ and let $\mathbb{D} := \{\mathbf{x} \equiv \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \in \mathbf{R}^2 : (x^1)^2 + (x^2)^2 < 1\}$ denote the unit disk on the plane. *The symmetric m-covariant tensor field* $\mathbf{a}^{(m)}(\mathbf{x})$ defined on $\mathbb{D}$ can be treated as a mapping

$$\mathbf{a}^{(m)} : \mathbb{D} \to \mathbf{S}_m,$$

where $\mathbf{S}_m$ denote the space of symmetric m-covariant tensors in $\mathbf{R^2}$ with the inner product $\langle \cdot, \cdot \rangle_{\mathbf{S}_m}$. Here and subsequently the superscript (m) is a remember of the rank of the tensor field $\mathbf{a}$.

So, for $m = 1$ we have $\mathbf{S}_1$ — the space of 1-covariant tensors, or simply (co)vectors,

$$\mathbf{S}_1 := \left\{ \mathbf{a}^{(1)} := \sum_{i=1}^{2} a_i dx^i \equiv \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \right\}, \ \langle \mathbf{a}^{(1)}, \mathbf{b}^{(1)} \rangle_{\mathbf{S}_1} := \sum_{i=1}^{2} a_i b_i \tag{1}$$

and for $m = 2$ we have $\mathbf{S}_2$ — the space of symmetric 2-covariant tensors, or simply 2-tensor,

$$\mathbf{S}_2 := \left\{ \mathbf{a}^{(2)} := \sum_{i,j=1}^{2} a_{ij} dx^i \otimes dx^j \equiv \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \ a_{ij} = a_{ji} \right\}, \ \langle \mathbf{a}^{(2)}, \mathbf{b}^{(2)} \rangle_{\mathbf{S}_2} := \sum_{i,j=1}^{2} a_{ij} b_{ij}. \quad (2)$$

First we recall the classical Radon transform $\mathcal{R}$ of m-tensor fields $\mathbf{a}^{(m)}$ supported on the unit disk $\mathbb{D}$

$$[\mathcal{R}\mathbf{a}^{(m)}](s, \varphi) := \int_{-w(s)}^{w(s)} \mathbf{a}^{(m)}(s\boldsymbol{\theta} + t\boldsymbol{\theta}^{\perp})dt, \ s \in [-1, 1], \ \varphi \in [0, 2\pi), \quad (3)$$

where $w(s) = (1 - s^2)^{1/2}$, $\boldsymbol{\theta} = (\cos\varphi, \sin\varphi)^\top$, $\boldsymbol{\theta}^\perp = (-\sin\varphi, \cos\varphi)^\top$.

We define *the m-tensor Radon transform $\mathcal{R}_m$* of m-tensor fields $\mathbf{a}^{(m)}$ supported on the unit disk $\mathbb{D}$ as follows

$$[\mathcal{R}_m \mathbf{a}^{(m)}](s, \varphi) := \langle (\boldsymbol{\theta}^\perp)^m, [\mathcal{R}\mathbf{a}^{(m)}](\varphi, s) \rangle_{\mathbf{S}_m}. \quad (4)$$

Here $\langle \cdot, \cdot \rangle_{\mathbf{S}_m}$ is the pointwise inner product (1) or (2), $(\boldsymbol{\theta}^\perp)^m$ is the "probe" vector/tensor, namely $(\boldsymbol{\theta}^\perp)^1 := \boldsymbol{\theta}^\perp$ and $(\boldsymbol{\theta}^\perp)^2 := \boldsymbol{\theta}^\perp \otimes \boldsymbol{\theta}^\perp$, $\otimes$ is the tensor product.

We will treat the transform (4) as operator $\mathcal{R}_m : L_2(\mathbb{D}, \mathbf{S}_m) \to L_2([-1, 1] \times [0, 2\pi), w^{-1})$, and it is known that Ker$\mathcal{R}_m$ coincides with the subspace of potential (irrotational) fields $\mathbf{d}\mathbf{H}_0^1(\mathbb{D}, \mathbf{S}_{m-1})$, so we can say that potential fields are "invisible" for the tensorial Radon transform $\mathcal{R}_m$.

The function $g(s, \varphi) = [\mathcal{R}_m \mathbf{a}^{(m)}](s, \varphi)$ is a tomographic date or *sinogram.* A sinogram is an image of the Radon transform (4), where s and φ form the vertical and horizontal axes respectively, of a Cartesian coordinate system. The test vector field and its sinogram are displayed in Figures 1(a) and (b) respectively. For a particular φ the function $g(s, \varphi)$ is a function of s and is called *a parallel ray projection* or just a projection. In the most real problems we expect to have a discrete version of a sinogram sampled for many values of s and φ.

Our task is to reconstruct solenoidal vector/2-tensor field $\mathbf{a}^{(m)}(\mathbf{x})$ from its known vector/2-tensor Radon transform (4), $g(s, \varphi) = [\mathcal{R}_m \mathbf{a}^{(m)}](s, \varphi)$.

2. PRELIMINARIES

In this section, we review some facts from vector and tensor analysis [17], [20], define some functional spaces of tensor fields and establish the notations that will be used later. At the end we introduce *Chebyshev bivariate ridge polynomials* and recall their application to a scalar Radon transform.

2.1. Vector and Tensor Analysis

We shall denote the class of real-valued m-covariant symmetric tensor fields $\mathbf{a}^{(m)}(\mathbf{x})$, whose all components are functions from $C^k(\mathbb{D})$, $1 \leq k \leq \infty$ by $\mathbf{C}^k(\mathbb{D}, \mathbf{S}_m)$. A subset of $\mathbf{C}^k(\mathbb{D}, \mathbf{S}_m)$ whose finite support is contained in $\mathbb{D}$ will be denoted by $\mathbf{C}_0^k(\mathbb{D}, \mathbf{S}_m)$. $\mathbf{L}_2(\mathbb{D}, \mathbf{S}_m)$ and Sobolev spaces $\mathbf{H}^k(\mathbb{D}, \mathbf{S}_m)$, $\mathbf{H}_0^1(\mathbb{D}, \mathbf{S}_m)$ are spaces of vector or tensor valued functions with components in $L_2(\mathbb{D})$ and $H^k(\mathbb{D})$, $H_0^1(\mathbb{D})$ respectively. The space $\mathbf{L}_2(\mathbb{D}, \mathbf{S}_m)$ is a Hilbert space with the inner product and the finite norm, denoted by $\langle\langle \cdot, \cdot \rangle\rangle$ and $|| \cdot ||$

$$\langle\langle \mathbf{a}^{(m)}, \ \mathbf{b}^{(m)} \rangle\rangle \equiv \langle\langle \mathbf{a}^{(m)}, \ \mathbf{b}^{(m)} \rangle\rangle_{\mathbf{L}_2(\mathbb{D}, \mathbf{S}_m)} := \iint_{\mathbb{D}} \langle \mathbf{a}^{(m)}(\mathbf{x}), \ \mathbf{b}^{(m)}(\mathbf{x}) \rangle_{\mathbf{S}_m} dx^1 dx^2,$$

$$||\mathbf{a}^{(m)}||^2 \equiv ||\mathbf{a}^{(m)}||^2_{\mathbf{L}_2(\mathbb{D}, \mathbf{S}_m)} := \langle\langle \mathbf{a}^{(m)}, \ \mathbf{a}^{(m)} \rangle\rangle.$$

We will always denote vector and tensor fields and any related quantities such as functional spaces, by boldface characters.

Now we recall the basic differential operation of vector and 2-tensor analysis and will discuss both vector ($m = 1$) and 2-tensor cases ($m = 2$) in a parallel manner. A smooth m-tensor field $\mathbf{a}^{(m)} \in \mathbf{C}^k(\mathbb{D}, \mathbf{S}_m)$ is called *solenoidal or divergence-free* if its divergence equals to zero, where the divergence δ of a smooth vector field $\mathbf{a}^{(1)}$ is define as

$$\delta \mathbf{a}^{(1)} \equiv \operatorname{div} \mathbf{a}^{(1)} = \operatorname{div} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} := \frac{\partial a_1}{\partial x^1} + \frac{\partial a_2}{\partial x^2}$$

and the divergence δ of a smooth 2-tenor field $\mathbf{a}^{(2)}$ is define as

$$\delta \, \mathbf{a}^{(2)} = \delta \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} := \begin{pmatrix} \dfrac{\partial a_{11}}{\partial x^1} + \dfrac{\partial a_{12}}{\partial x^2} \\ \\ \dfrac{\partial a_{21}}{\partial x^1} + \dfrac{\partial a_{22}}{\partial x^2} \end{pmatrix}, \ a_{12} = a_{21}.$$

In connection with the Helmholtz-Hodge orthogonal decomposition of the space $\mathbf{L}_2(\mathbb{D}, \mathbf{S}_m)$, corresponding to the operators d and δ, we define two subspaces of solenoidal or divergence-free m-tensor fields, the first is

$$\mathbf{H}(\mathbb{D}, \mathbf{S}_m;\ \delta = \mathbf{0}) := \{\mathbf{a}^{(m)} \in \mathbf{H}(\mathbb{D}, \mathbf{S}_m;\ \delta)\Big|\ \delta\, \mathbf{a}^{(m)} = \mathbf{0}\}, \tag{5}$$

where $H(\mathbb{D}, \mathbf{S}_m; \delta)$ is the graph space of $\delta-$operator over $\mathbf{L}_2(\mathbb{D}, \mathbf{S}_m)$, i.e.

$$H(\mathbb{D}, \mathbf{S}_m; \delta) := \{\mathbf{a}^{(m)} \in \mathbf{L}_2(\mathbb{D}, \mathbf{S}_m)\Big|\delta\mathbf{a}^{(m)} \in \mathbf{L}_2(\mathbb{D}, \mathbf{S}_{m-1})\}. \tag{6}$$

It will be a Hilbert space under the graph norm. It is clear that subspace $\mathbf{H}(\mathbb{D}, \mathbf{S}_m;\ \delta = \mathbf{0})$ is a completion (closure) of the set of smooth solenoidal m-tensor fields with respect to the norm $||\cdot||$ of $\mathbf{L}_2(\mathbb{D}, \mathbf{S}_m)$.

The second subspace is $\mathbf{H}_0(\mathbb{D}, \mathbf{S}_m;\ \delta = \mathbf{0})$, which consists of all solenoidal tensor fields on $\mathbb{D}$ that are tangent to $\partial\mathbb{D}$

$$\mathbf{H}_0(\mathbb{D}, \mathbf{S}_m;\ \delta = \mathbf{0}) := \{\mathbf{a}^{(m)} \in \mathbf{H}(\mathbb{D}, \mathbf{S}_m; \delta = 0)\Big|\ \langle(\mathbf{n})^m, \mathbf{a}^{(m)}\rangle_{\mathbf{S}_m} = 0 \text{ on } \partial\mathbb{D}\}, \tag{7}$$

where $\mathbf{n}$ being the unit outward normal to the boundary of $\mathbb{D}$, $(\mathbf{n})^1 := \mathbf{n},\ (\mathbf{n})^2 := \mathbf{n} \otimes \mathbf{n}$. Vector fields from subspace $\mathbf{H}_0(\mathbb{D}, \mathbf{S}_1;\ \delta = \mathbf{0})$ are used to represent incompressible fluid flows within fixed boundaries, and magnetic fields inside plasma containment device.

A m-tensor field $\mathbf{a}^{(m)} \in \mathbf{C}^\infty(\mathbb{D}, \mathbf{S}_m)$ is called a smooth *potential* tensor field, if for some $(m-1)$-tensor field $\mathbf{v}^{(m-1)} \in \mathbf{C}^\infty(\mathbb{D}, \mathbf{S}_{m-1})$ we have $\mathbf{a}^{(m)} = \mathrm{d}\mathbf{v}^{(m-1)}$ and $\mathbf{v}^{(m-1)}$ is the potential, where d is *the symmetrized covariant derivative* of the symmetric $(m-1)$-tensor $\mathbf{v}^{(\mathbf{m-1})}$. For $m = 1$ we have

$$\mathrm{d}v \equiv \mathrm{grad}\ v := \left(\frac{\partial v}{\partial x^1}, \frac{\partial v}{\partial x^2}\right)^\top,$$

v is a scalar potential. In fluid mechanics this vector fields are called *irrotational vector fields.*

For $m = 2$ we have

$$\mathrm{d}\mathbf{v}^{(1)} = \mathrm{d}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} := \begin{pmatrix} \dfrac{\partial v_1}{\partial x^1} & \dfrac{1}{2}\left(\dfrac{\partial v_1}{\partial x^2} + \dfrac{\partial v_2}{\partial x^1}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial v_1}{\partial x^2} + \dfrac{\partial v_2}{\partial x^1}\right) & \dfrac{\partial v_2}{\partial x^2} \end{pmatrix},$$

$\mathbf{v}^{(1)}$ is a (co)vector potential and in this case the symmetrized gradient of a (co)vector field is a symmetric 2-tensor field.

We consider three subspaces of potential m-tensor fields, $\mathrm{d}\mathbf{H}^1(\mathbb{D}, \mathbf{S}_{m-1})$ (*gradient* fields), $\mathrm{d}\mathbf{H}^1_0(\mathbb{D}, \mathbf{S}_{m-1})$ (*grounded gradient* fields) and $\mathbf{HG}$ (*harmonic gradient* fields), define by

$$\begin{aligned} \mathrm{d}\mathbf{H}^1(\mathbb{D}, \mathbf{S}_{m-1}) &:= \{\mathrm{d}\mathbf{v}^{(m-1)}\Big|\mathbf{v}^{(m-1)} \in \mathbf{H}^1(\mathbb{D}, \mathbf{S}_{m-1})\}, \\ \mathrm{d}\mathbf{H}^1_0(\mathbb{D}, \mathbf{S}_{m-1}) &:= \{\mathrm{d}\mathbf{v}^{(m-1)}\Big|\mathbf{v}^{(m-1)} \in \mathbf{H}^1_0(\mathbb{D}, \mathbf{S}_{m-1})\}, \\ \mathbf{HG} &:= \mathrm{d}\mathbf{H}^1(\mathbb{D}, \mathbf{S}_{m-1}) \ominus \mathrm{d}\mathbf{H}^1_0(\mathbb{D}, \mathbf{S}_{m-1}), \end{aligned}$$

where $\ominus$ is the orthogonal subtraction of subspaces.

It is known, that a m-tensor field can be represented as a sum of the divergence-free (solenoidal) component and symmetrized gradient of a $(m-1)$-tensor potential. The classical result in this direction belongs to H. Weyl [21] and is connected with the decomposition of the $\mathbf{L}_2$ space of vector fields into the orthogonal sum of solenoidal and potential fields. The following theorem is about the orthogonal decompositions in $\mathbf{L}_2(\mathbb{D}, \mathbf{S}_2)$, which are often referred to as *Helmholtz–Hodge decomposition*:

Theorem. *The space* $\mathbf{L}_2(\mathbb{D}, \mathbf{S}_m)$ *is the direct sum of three mutually orthogonal subspaces*

$$\mathbf{L}_2(\mathbb{D}, \mathbf{S}_m) = \mathbf{H}_0(\mathbb{D}, \mathbf{S}_m; \delta = \mathbf{0}) \oplus \mathbf{HG} \oplus \mathrm{d}\mathbf{H}^1_0(\mathbb{D}, \mathbf{S}_{m-1}) \tag{8}$$

with Ker $\delta = \mathbf{H}(\mathbb{D}, \mathbf{S}_m; \delta = \mathbf{0}) = \mathbf{H}_0(\mathbb{D}, \mathbf{S}_m; \delta = 0) \oplus \mathbf{HG}$.

Form this decomposition we see, that the subspace of harmonic gradient $\mathbf{HG}$ may be defined as

$$\mathbf{HG} = \mathbf{H}(\mathbb{D}, \mathbf{S}_m; \delta = \mathbf{0}) \cap \mathrm{d}\mathbf{H}^1(\mathbb{D}, \mathbf{S}_{m-1}).$$

In fluid mechanics the harmonic gradient component of vector field, which both solenoidal and irrotational, is called also as *a laminar component.*

The survey of the historical threads that led to the Helmholtz-Hodge decomposition theorem and the proof for 3D case may be found in [4]. See also [5], [7], [8], [17] regarding this problem.

2.2. Chebyshev bivariate ridge polynomials and their Radon transform

We shall use the Chebyshev polynomials of the first and second kind in the interval $t \in [-1,1]$

$$T_n(t) := \cos(n \arccos t), \quad U_n(t) := \frac{\sin((n+1)\arccos t)}{\sqrt{1-t^2}}, \; n = 0, 1, \dots \, . \tag{9}$$

Let's denote the bivariate Chebyshev polynomials by

$$U_{nk}(\mathbf{x}) := \frac{(-1)^k}{\sqrt{\pi}} U_n \left(x^1 \cos \frac{\pi k}{n+1} + x^2 \sin \frac{\pi k}{n+1} \right), \; n = 0, 1, \dots , \; k = 0, 1, \dots, n. \tag{10}$$

Each function in (10) is *a plane wave* propagating in the direction $\left(\cos \frac{\pi k}{n+1}, \sin \frac{\pi k}{n+1} \right)^\top$, or *a ridge functions* of $\mathbf{x}$, and the univariate function U_n is called the profile of U_{nk}. The system (10) is a complete orthonormal system in $L_2(\mathbb{D})$, see [2], [14], [23].

Let us denote the Radon transforms of basis functions (10) by $u_{nk}(s,\varphi)$,

$$u_{nk}(s,\varphi) := [\mathcal{R}U_{nk}]\,(s,\varphi).$$

We have, see for example [2], [10], [14], that

$$u_{nk}(s,\varphi) = \frac{2(-1)^k}{\sqrt{\pi}(n+1)} U_n \left[\cos \left(\varphi - \frac{\pi k}{n+1} \right) \right] \sqrt{1-s^2} U_n(s) \tag{11}$$

and the functions u_{nk} are orthogonal in the weighted Hilbert space $L_2([-1,1] \times [0,2\pi), w^{-1})$ with the weight function $w^{-1} = 1/\sqrt{1-s^2}$.

So, if some function $a(\mathbf{x}) \in L_2(\mathbb{D})$ then it is represented by its Fourier series

$$a(\mathbf{x}) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{nk} U_{nk}(\mathbf{x}),$$

and its Radon transform will look like

$$g(s,\varphi) := [\mathcal{R}a]\,(s,\varphi) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{nk} u_{nk}(s,\varphi).$$

One of the ways for calculating coefficients a_{nk} is the following: let us introduce the n-th Chebyshev momentum $g_n(\varphi)$ of $g(s,\varphi)$

$$g_n(\varphi) := \int_{-1}^{1} g(s,\varphi) U_n(s) ds = \sqrt{\pi} \sum_{k=0}^{n} \frac{(-1)^k}{n+1} a_{nk} U_n \left(\cos \left(\varphi - \frac{\pi k}{n+1} \right) \right). \tag{12}$$

Using the substitution $\varphi = \frac{\pi k}{n+1}$ for $n = 0, 1, \dots; \; k = 0, 1, \dots, n$ in (12), we get $a_{nk} = \frac{(-1)^k}{\sqrt{\pi}} g_n \left(\frac{\pi k}{n+1} \right)$, which leads to the inversion formula

$$a(\mathbf{x}) = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k g_n \left(\frac{\pi k}{n+1} \right) U_{nk}(\mathbf{x}).$$

Here only the countably number of parallel ray projections $g\left(\bullet, \frac{\pi k}{n+1} \right)$ is needed to recover the unknown function $a(\mathbf{x})$.

3. THE POLYNOMIAL BASES FOR $\mathbf{L}_2(\mathbb{D}, \mathbf{S}_m)$ AND ITS SUBSPACES

In this section we construct the orthogonal polynomial bases for the space of tensor fields $\mathbf{L}_2(\mathbb{D}, \mathbf{S}_m)$ and for the various subspaces that appear in the Helmholtz-Hodge decomposition theorem both for $m = 1, 2$. We should first of all introduce for $n = 0, 1, 2, ...$ and $k = 0, 1, 2, ..., n$ the unit vectors defined by

$$\boldsymbol{\theta}_{nk}^{(1),1} := \left(-\sin\frac{\pi k}{n+1}, \cos\frac{\pi k}{n+1}\right)^{\top}, \ \boldsymbol{\theta}_{nk}^{(1),2} := \left(\cos\frac{\pi k}{n+1}, \sin\frac{\pi k}{n+1}\right)^{\top}, \ (\boldsymbol{\theta}_{nk}^{(1),2})^{\perp} = \boldsymbol{\theta}_{nk}^{(1),1}.$$

Proposition 1. *The system of vector fields* $\{\mathbf{U}_{nk}^{(1),0}, \mathbf{U}_{nk}^{(1),1}\}$ *defined by*

$$\mathbf{U}_{nk}^{(1),0}(\mathbf{x}) := \frac{(-1)^k}{\sqrt{\pi}}\boldsymbol{\theta}_{nk}^{(1),1} U_{nk}(\mathbf{x}), \tag{13}$$

$$\mathbf{U}_{nk}^{(1),1}(\mathbf{x}) := \frac{(-1)^k}{\sqrt{\pi}}\boldsymbol{\theta}_{nk}^{(1),2} U_{nk}(\mathbf{x}) \tag{14}$$

is an orthonormal polynomial basis for the space $\mathbf{L}_2(\mathbb{D}, \mathbf{S}_1)$. *Moreover* $\mathbf{U}_{nk}^{(1),0}$ *— solenoidal,* $\operatorname{div}\mathbf{U}_{nk}^{(1),0} = 0$, *but* $\mathbf{U}_{nk}^{(1),1}$ *are potential vector fields.*

Let's define new polynomial vector fields by

$$\widetilde{\mathbf{U}}_{nk}^{(1),0} := \mathbf{U}_{nk}^{(1),0} - \mathbf{V}_{n,n+1}^{(1)}, \tag{15}$$

$$\widetilde{\mathbf{U}}_{nk}^{(1),1} := \mathbf{U}_{nk}^{(1),1} - \mathbf{W}_{n,n+1}^{(1)}, \tag{16}$$

$$\mathbf{V}_{n,n+1}^{(1)} := \frac{1}{n+1}\sum_{k=0}^{n}\mathbf{U}_{nk}^{(1),0}, \tag{17}$$

$$\mathbf{W}_{n,n+1}^{(1)} := \frac{1}{n+1}\sum_{k=0}^{n}\mathbf{U}_{nk}^{(1),1}. \tag{18}$$

We see, that

$$\widetilde{\mathbf{U}}_{00}^{(1),0} = \widetilde{\mathbf{U}}_{00}^{(1),1} = 0, \ \mathbf{V}_{01}^{(1)} = \mathbf{U}_{00}^{(1),0}, \ \mathbf{W}_{01}^{(1)} = \mathbf{U}_{00}^{(1),1}.$$

Proposition 2. *For vector fields (15)–(18) the following statements hold:*

(a) $\overline{\mathbf{Span}}\left\{\widetilde{\mathbf{U}}_{nk}^{(1),0}\right\} = \mathbf{H}_0(\mathbb{D}, \mathbf{S}_1; \operatorname{div} = 0)$

(b) $\{\mathbf{V}_{n,n+1}^{(1)}, \mathbf{W}_{n,n+1}^{(1)}\}$ *form an orthogonal basis for* $\mathbf{HG}$

(c) $\overline{\mathbf{Span}}\left\{\widetilde{\mathbf{U}}_{nk}^{(1),1}\right\} = \operatorname{grad} H_0^1(\mathbb{D})$

(d) $\{\mathbf{U}_{nk}^{(1),1}, \mathbf{V}_{n,n+1}^{(1)}\}$ *form an orthogonal basis for* $\operatorname{grad} H^1(\mathbb{D})$

(e) $\{\mathbf{U}_{nk}^{(1),0}, \mathbf{W}_{n,n+1}^{(1)}\}$ *form an orthogonal basis for* $\mathbf{H}(\mathbb{D}, \mathbf{S}_1; \operatorname{div} = 0) = \mathbf{H}_0(\mathbb{D}, \mathbf{S}_1; \operatorname{div} = 0) \oplus \mathbf{HG}$.

In this proposition the set of finite linear combinations of elements v_k is denoted by $\mathbf{Span}\{v_k\}$ and $\overline{\mathbf{Span}}\{v_k\}$ denotes the norm closure.

By analogy, we formulate propositions for the 2-tensor case.

Proposition 3. *The system of tensor fields* $\{\mathbf{U}_{nk}^{(2),0}, \mathbf{U}_{nk}^{(2),1}, \mathbf{U}_{nk}^{(2),2}\}$ *defined by*

$$\mathbf{U}_{nk}^{(2),0}(\mathbf{x}) := \frac{(-1)^k}{\sqrt{\pi}}\boldsymbol{\theta}_{nk}^{(1),1} \otimes^s \boldsymbol{\theta}_{nk}^{(1),1} U_{nk}(\mathbf{x}), \tag{19}$$

$$\mathbf{U}_{nk}^{(2),1}(\mathbf{x}) := \frac{(-1)^k}{\sqrt{\pi}}\boldsymbol{\theta}_{nk}^{(1),1} \otimes^s \boldsymbol{\theta}_{nk}^{(1),2} U_{nk}(\mathbf{x}), \tag{20}$$

$$\mathbf{U}_{nk}^{(2),2}(\mathbf{x}) := \frac{(-1)^k}{\sqrt{\pi}}\boldsymbol{\theta}_{nk}^{(1),2} \otimes^s \boldsymbol{\theta}_{nk}^{(1),2} U_{nk}(\mathbf{x}) \tag{21}$$

is orthogonal polynomial basis for $\mathbf{L}_2(\mathbb{D}, \mathbf{S}_2)$ *with* $||\mathbf{U}_{nk}^{(2),0}|| = 1$, $||\mathbf{U}_{nk}^{(2),1}|| = \frac{1}{\sqrt{2}}$, $||\mathbf{U}_{nk}^{(2),2}|| = 1$. *Moreover* $\mathbf{U}_{nk}^{(2),0}$ *— solenoidal,* $\delta\mathbf{U}_{nk}^{(2),0} = 0$, *but* $\mathbf{U}_{nk}^{(2),1}$ *and* $\mathbf{U}_{nk}^{(2),2}$ *are potential tensor fields.*

In this proposition we used the operation $\otimes^s$, which denotes *the symmetrized tensor product* of two vectors, i.e. if $\mathbf{a}^{(1)}$ and $\mathbf{b}^{(1)}$ are some vectors, then

$$\mathbf{a}^{(1)} \otimes^s \mathbf{b}^{(1)} := \frac{1}{2}\left(\mathbf{a}^{(1)} \otimes \mathbf{b}^{(1)} + \mathbf{b}^{(1)} \otimes \mathbf{a}^{(1)}\right).$$

Let's define new polynomial 2-tensor fields by

$$\widetilde{\mathbf{U}}_{00}^{(2),0} := 0, \tag{22}$$

$$\widetilde{\mathbf{U}}_{nk}^{(2),0} := \mathbf{U}_{nk}^{(2),0} - 2\cos\frac{\pi k}{n+1}\mathbf{V}_{n,n+1}^{(2)} - 2\sin\frac{\pi k}{n+1}\mathbf{V}_{n,n+2}^{(2)} \ (n \geq 1), \tag{23}$$

$$\widetilde{\mathbf{U}}_{00}^{(2),1} := 0, \tag{24}$$

$$\widetilde{\mathbf{U}}_{nk}^{(2),1} := \mathbf{U}_{nk}^{(2),1} + \frac{2}{3}\sin\frac{\pi k}{n+1}\mathbf{W}_{n,n+1}^{(2)} - \frac{2}{3}\cos\frac{\pi k}{n+1}\mathbf{W}_{n,n+2}^{(2)} \quad (n \geq 1), \tag{25}$$

$$\widetilde{\mathbf{U}}_{00}^{(2),2} := 0, \tag{26}$$

$$\widetilde{\mathbf{U}}_{nk}^{(2),2} := \mathbf{U}_{nk}^{(2),2} - \frac{2}{3}\cos\frac{\pi k}{n+1}\mathbf{W}_{n,n+1}^{(2)} - \frac{2}{3}\sin\frac{\pi k}{n+1}\mathbf{W}_{n,n+2}^{(2)} \quad (n \geq 1), \tag{27}$$

$$\mathbf{V}_{01}^{(2)} := \mathbf{U}_{00}^{(2),0}, \ \mathbf{V}_{n,n+1}^{(2)} := \frac{1}{n+1}\sum_{k=0}^{n}\cos\frac{\pi k}{n+1}\mathbf{U}_{nk}^{(2),0}, \tag{28}$$

$$\mathbf{V}_{n,n+2}^{(2)} := \frac{1}{n+1}\sum_{k=0}^{n}\sin\frac{\pi k}{n+1}\mathbf{U}_{nk}^{(2),0}, \tag{29}$$

$$\mathbf{W}_{01}^{(2)} := \mathbf{U}_{00}^{(2),2}, \ \mathbf{W}_{n,n+1}^{(2)} := \frac{1}{n+1}\sum_{k=0}^{n}\left(-2\sin\frac{\pi k}{n+1}\mathbf{U}_{nk}^{(2),1} + \cos\frac{\pi k}{n+1}\mathbf{U}_{nk}^{(2),2}\right) \quad (n \geq 1), \tag{30}$$

$$\mathbf{W}_{02}^{(2)} := \mathbf{U}_{00}^{(2),1}, \ \mathbf{W}_{n,n+2}^{(2)} := \frac{1}{n+1}\sum_{k=0}^{n}\left(\ 2\cos\frac{\pi k}{n+1}\mathbf{U}_{nk}^{(2),1} + \sin\frac{\pi k}{n+1}\mathbf{U}_{nk}^{(2),2}\right) \quad (n \geq 1). \tag{31}$$

Proposition 4. *For 2-tensor fields (22)–(31) the following statements hold:*

(a) $\overline{\mathbf{Span}}\left\{\widetilde{\mathbf{U}}_{nk}^{(2),0}\right\} = \mathbf{H}_0(\mathbb{D}, \mathbf{S}_2; \delta = \mathbf{0})$

(b) $\{\mathbf{V}_{n,n+1}^{(2)}, \mathbf{V}_{n,n+2}^{(2)}, \mathbf{W}_{n,n+1}^{(2)}, \mathbf{W}_{n,n+2}^{(2)}\}$ *form an orthogonal basis for* $\mathbf{HG}$

(c) $\overline{\mathbf{Span}}\left\{\widetilde{\mathbf{U}}_{nk}^{(2),1}, \ \widetilde{\mathbf{U}}_{nk}^{(2),2}\right\} = \mathrm{d}\mathbf{H}_0^1(\mathbb{D}, \mathbf{S}_1)$

(d) $\{\mathbf{U}_{nk}^{(2),1}, \mathbf{U}_{nk}^{(2),2}, \mathbf{V}_{n,n+1}^{(2)}, \mathbf{V}_{n,n+2}^{(2)}\}$ *form an orthogonal basis for* $\mathrm{d}\mathbf{H}^1(\mathbb{D}, \mathbf{S}_1)$

(e) $\{\mathbf{U}_{nk}^{(2),0}, \mathbf{W}_{n,n+1}^{(2)}, \mathbf{W}_{n,n+2}^{(2)}\}$ *form an orthogonal basis for* $\mathbf{H}(\mathbb{D}, \mathbf{S}_2; \delta = \mathbf{0}) = \mathbf{H}_0(\mathbb{D}, \mathbf{S}_2; \delta = \mathbf{0}) \oplus \mathbf{HG}$.

4. SERIES INVERSION OF THE TENSOR RADON TRANSFORM

We recall that only the solenoidal part of a vector or 2-tensor field can be reconstructed from the vector or tensor Radon transform, so the total tensor field is assumed to be solenoidal and is denoted by $\mathbf{a}_{sol}^{(m)}$. After we constructed polynomial bases for $\mathbf{H}(\mathbb{D}, \mathbf{S}_m; \delta = \mathbf{0})$, see Propositions 2(*e*), 4(*e*), we can give a series inversion for tensor tomography problem.

Theorem 1 (vector case, $m = 1$**).** *A solenoidal vector field* $\mathbf{a}_{sol}^{(1)} \in \mathbf{H}(\mathbb{D}, \mathbf{S}_1; \delta = 0)$ *and its Radon transform* $g(s, \varphi) := [\mathcal{R}_1 \mathbf{a}_{sol}^{(1)}](s, \varphi)$ *have the Fourier series expansions*

$$\mathbf{a}_{sol}^{(1)}(\mathbf{x}) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_{nk}\mathbf{U}_{nk}^{(1),0}(\mathbf{x}) + \sum_{n=0}^{\infty} a_{n,n+1}\mathbf{W}_{n,n+1}^{(1)}(\mathbf{x}),$$

$$g(s,\varphi) = -\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_{nk}\cos\left(\varphi - \frac{\pi k}{n+1}\right)u_{nk}(s,\varphi) + \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{a_{n,n+1}}{n+1}\sin[(n+1)\varphi]\sqrt{1-s^2}U_n(s)$$

and the following inversion formulae hold for the coefficients (n=0,1,...; k=0,...,n)

$$a_{nk} = -\frac{(-1)^k}{\sqrt{\pi}} g_n\left(\frac{\pi k}{n+1}\right),$$

$$a_{n,n+1} = \frac{n+1}{\sqrt{\pi}}\left(g_n\left(\frac{\pi/2}{n+1}\right) - \widetilde{g}_n\left(\frac{\pi/2}{n+1}\right)\right),$$

where $g_n(\varphi) := \int_{-1}^{1} g(s,\varphi)U_n(s)ds$ *is the n-th Chebyshev moment of* $g(s,\varphi)$ *and*

$$\widetilde{g}_n(\varphi) := \frac{1}{n+1}\sum_{k=0}^{n} g_n\left(\frac{\pi k}{n+1}\right)\cos\left(\varphi - \frac{\pi k}{n+1}\right)U_n\left(\cos\left(\varphi - \frac{\pi k}{n+1}\right)\right).$$

Theorem 2 (tensor case, $m = 2$). *A solenoidal 2-tensor field* $\mathbf{a}^{(2)}_{sol}(\mathbf{x}) \in \mathbf{H}(\mathbb{D}, \mathbf{S}_2; \delta = \mathbf{0})$ *and its Radon transform* $g(s,\varphi) := \left[\mathcal{R}_2\mathbf{a}^{(2)}_{sol}\right](s,\varphi)$ *have Fourier series expansions*

$$\mathbf{a}^{(2)}_{sol}(\mathbf{x}) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_{nk}\mathbf{U}^{(2),0}_{nk}(\mathbf{x}) + \sum_{n=0}^{\infty}\left(a_{n,n+1}\mathbf{W}^{(2)}_{n,n+1}(\mathbf{x}) + a_{n,n+2}\mathbf{W}^{(2)}_{n,n+2}(\mathbf{x})\right),$$

$$g(s,\varphi) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_{nk}\cos^2\left(\varphi - \frac{\pi k}{n+1}\right)u_{nk}(s,\varphi) + \frac{2}{\sqrt{\pi}}(a_{01}\sin\varphi - a_{02}\cos\varphi)\sin\varphi\sqrt{1-s^2}$$

$$+\frac{3}{\sqrt{\pi}}\sum_{n=1}^{\infty}\frac{1}{n+1}(a_{n,n+1}\sin\varphi - a_{n,n+2}\cos\varphi)\sin[(n+1)\varphi]\sqrt{1-s^2}U_n(s)$$

and the following inversion formulae hold for the coefficients (n=0,1,...; k=0,...,n)

$$a_{nk} = \frac{(-1)^k}{\sqrt{\pi}} g_n\left(\frac{\pi k}{n+1}\right),$$

$$(a_{01}\sin\varphi - a_{02}\cos\varphi)\sin\varphi = \frac{1}{\sqrt{\pi}}(g_0(\varphi) - \widetilde{g}_0(\varphi)), \tag{32}$$

$$(a_{n,n+1}\sin\varphi - a_{n,n+2}\cos\varphi)\sin[(n+1)\varphi] = \frac{2(n+1)}{3\sqrt{\pi}}(g_n(\varphi) - \widetilde{g}_n(\varphi))\ (n \geq 1), \tag{33}$$

where $g_n(\varphi) := \int_{-1}^{1} g(s,\varphi)U_n(s)ds$ *is the n-th Chebyshev moment of* $g(s,\varphi)$ *and*

$$\widetilde{g}_n(\varphi) := \frac{1}{n+1}\sum_{k=0}^{n} g_n\left(\frac{\pi k}{n+1}\right)\cos^2\left(\varphi - \frac{\pi k}{n+1}\right)U_n\left(\cos\left(\varphi - \frac{\pi k}{n+1}\right)\right).$$

Remark. The coefficients $a_{n,n+1}, a_{n,n+2}$ may be found from (32) and (33) in different ways, using only the countably number of parallel ray projections.

5. NUMERICAL RESULTS

The novel inversion algorithm, based on Theorems 1, 2 was implemented for vector and tensor cases. The following examples show some numerical results.

Example 1 (vector case, $m = 1$). Consider a vector field (see Figure 1), whose components are polynomials of degree 25 with random coefficients (a). Its longitudinal vector Radon transform $\mathcal{R}_1$ (sinogram) is shown in (b). Inverting the vectorial Radon transform using the Theorem 1 we reconstruct the solenoidal and harmonic parts (c) from $\mathbf{H}(\mathbb{D}, \mathbf{S}_1; \delta = 0) = \mathbf{H}_0(\mathbb{D}, \mathbf{S}_1; \delta = 0) \oplus \mathbf{HG}$. Then, knowing the structure of the basis, we can split the reconstructed field into the pure solenoidal part (d) from $\mathbf{H}_0(\mathbb{D}, \mathbf{S}_1; \delta = 0)$ and the harmonic part (f) from $\mathbf{HG}$. The vector field shown in (e) is the difference between the original vector field (a) and a reconstructed vector field (c) and represents the leftover potential part of the vector field (a) from $d\mathbf{H}^1_0(\mathbb{D}, \mathbf{S}_0)$, that is "invisible" for the vector Radon transform $\mathcal{R}_1$.

Example 2 (tensor case, $m = 2$). Consider a *solenoidal* tensor field (see Figure 2) from $\mathbf{H}(\mathbb{D}, \mathbf{S}_2; \delta = \mathbf{0})$, whose components (a_1), (a_2), (a_3) are polynomials of degree 40 with random coefficients. Inverting the tensor Radon transform $\mathcal{R}_2$ using the Theorem 2 we reconstructed it as solenoidal tensor fields whose component are polynomials of degree 20 (see (b_1), (b_2), (b_3)), degree 30 (see (c_1), (c_2), (c_3)) and degree 40 (see (d_1), (d_2), (d_3)). The last reconstructed tensor field (with components (d_1), (d_2), (d_3)) is almost identical to the original solenoidal tensor field.

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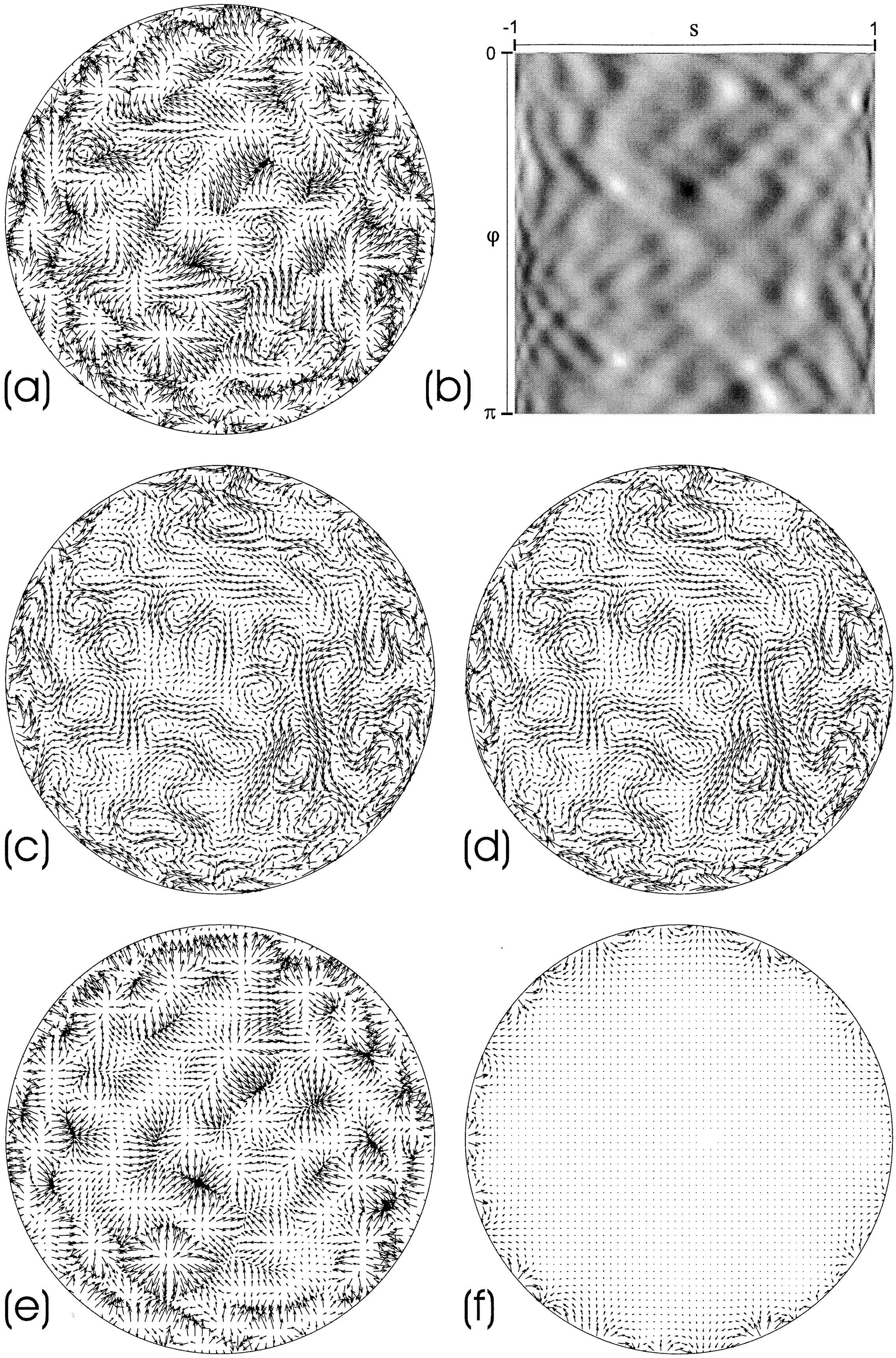

Figure 1: Illustration to the Example 1, see the text for details.

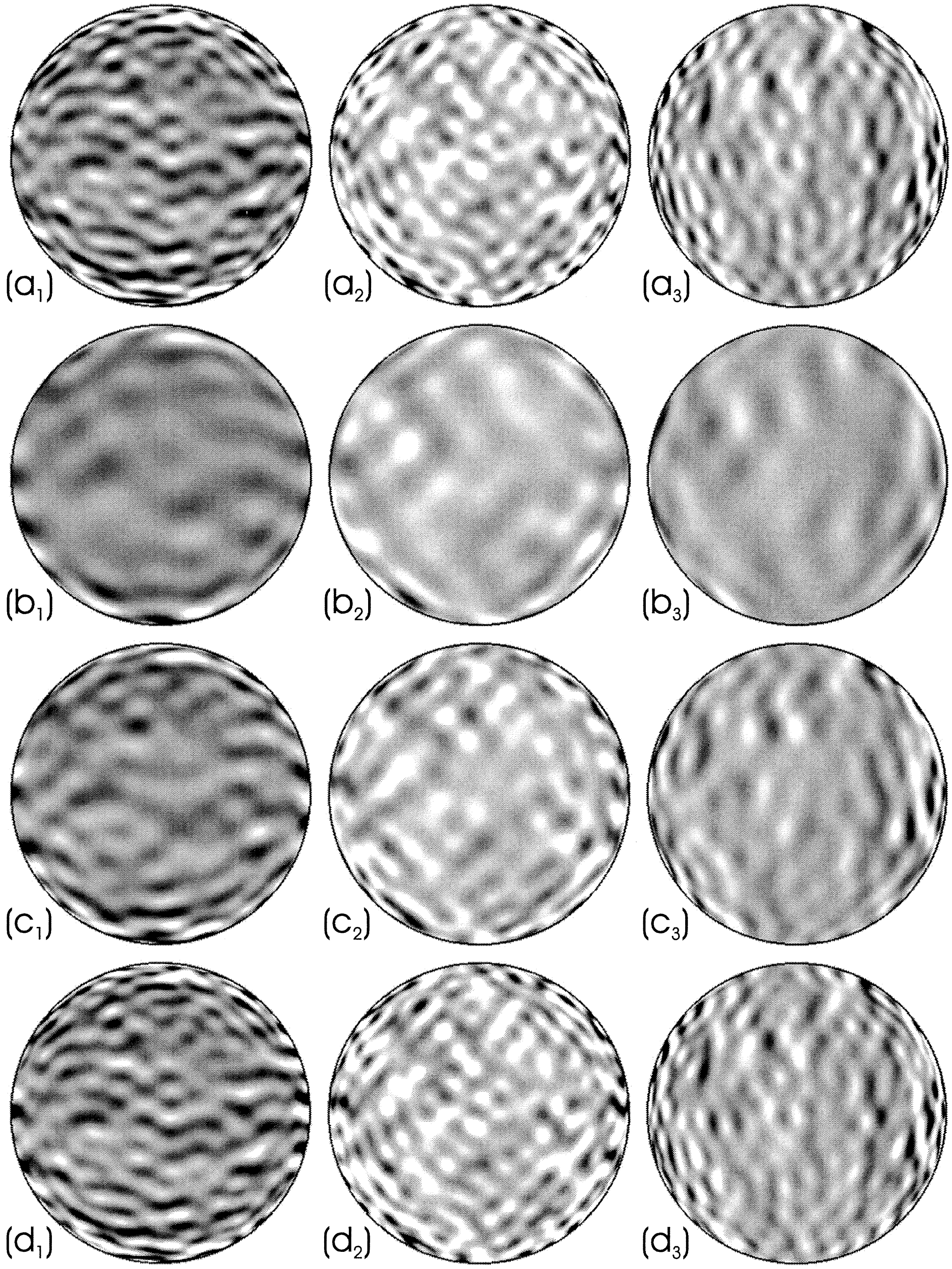

Figure 2: Illustration to the Example 2, see the text for details.

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